SBP Finite Differences

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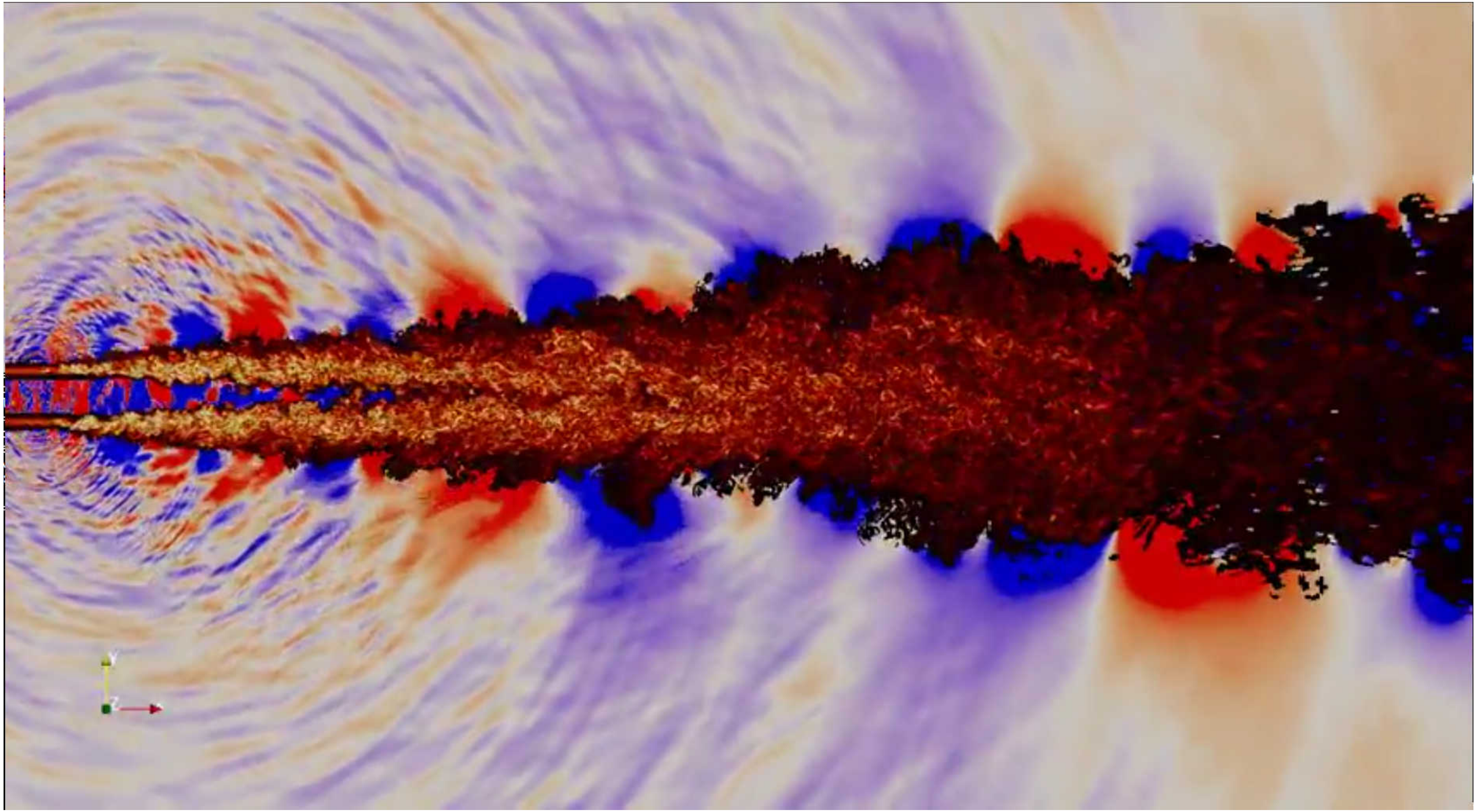
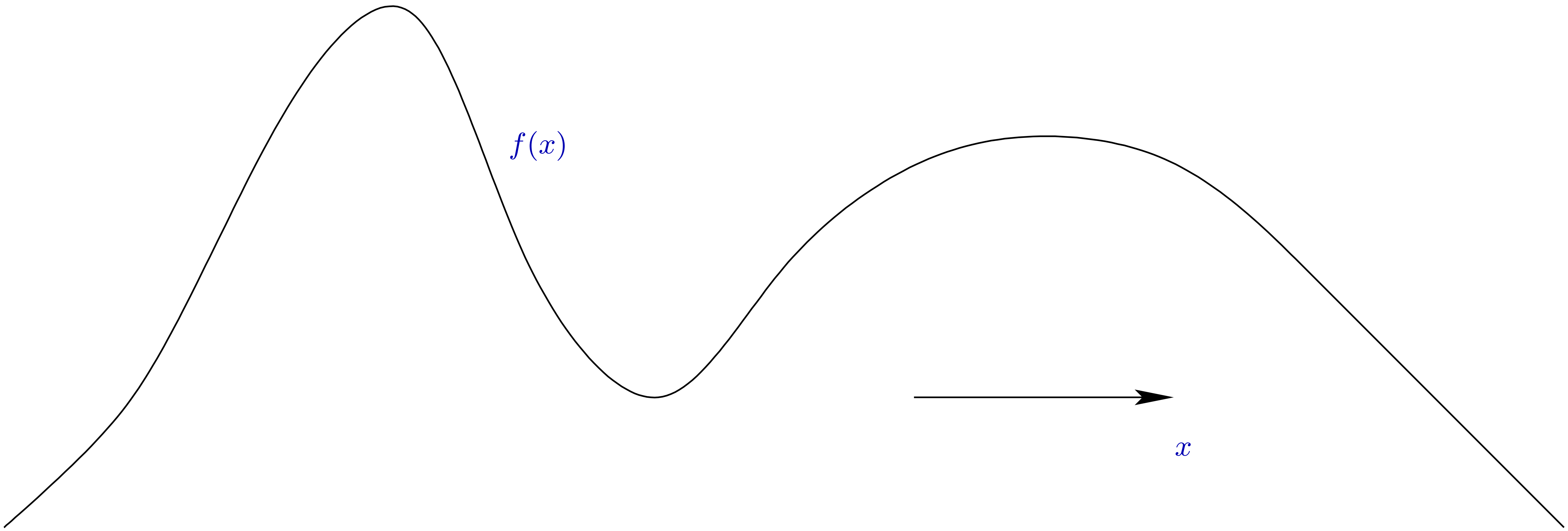
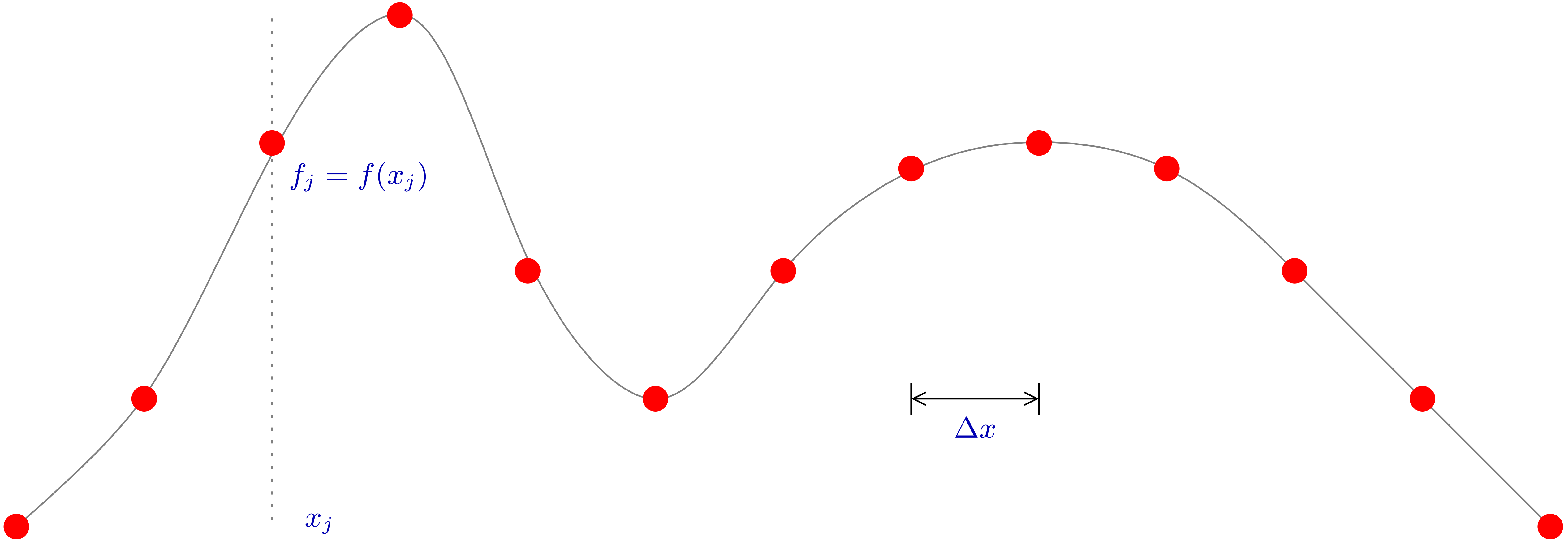
Sound from a jet



Finite Differences



Finite Differences



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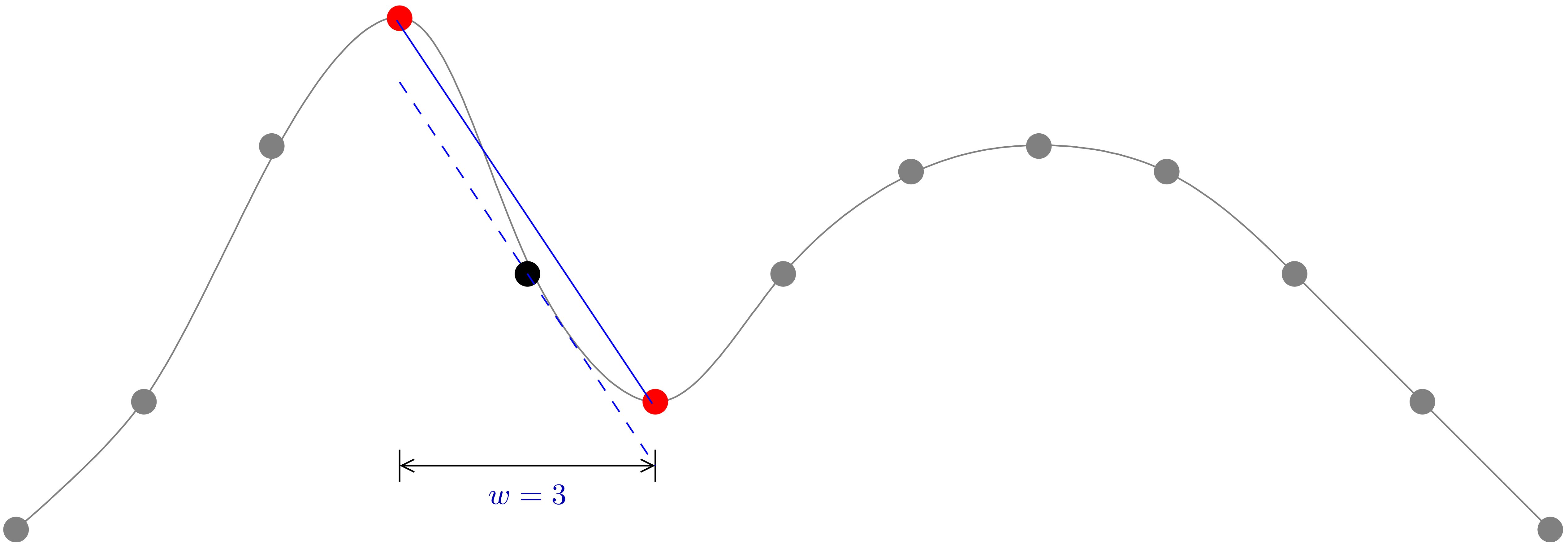

Could try:

$$f_j' = \frac{1}{\Delta x}(f_{j+1} - f_j) = f'(x_j) + O(\Delta x).$$

Better:

$$f_j' = \frac{1}{\Delta x}\left(-\frac{1}{2}f_{j-1} + \frac{1}{2}f_{j+1}\right) = f'(x_j) + O(\Delta x^2).$$

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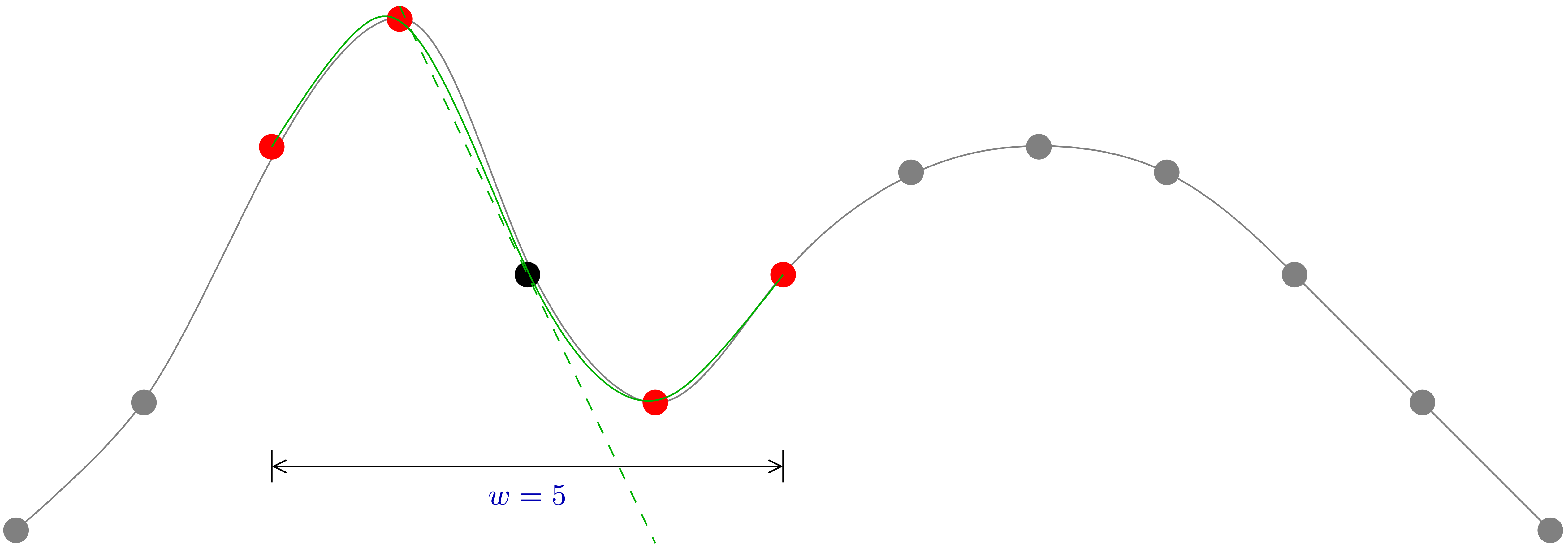

Even better:

$$f'_j = \frac{1}{\Delta x}\left(\frac{1}{12}f_{j-2} - \frac{2}{3}f_{j-1} + \frac{2}{3}f_{j+1} - \frac{1}{12}f_{j+2}\right) = f'(x_j) + O\big(\Delta x^4\big).$$

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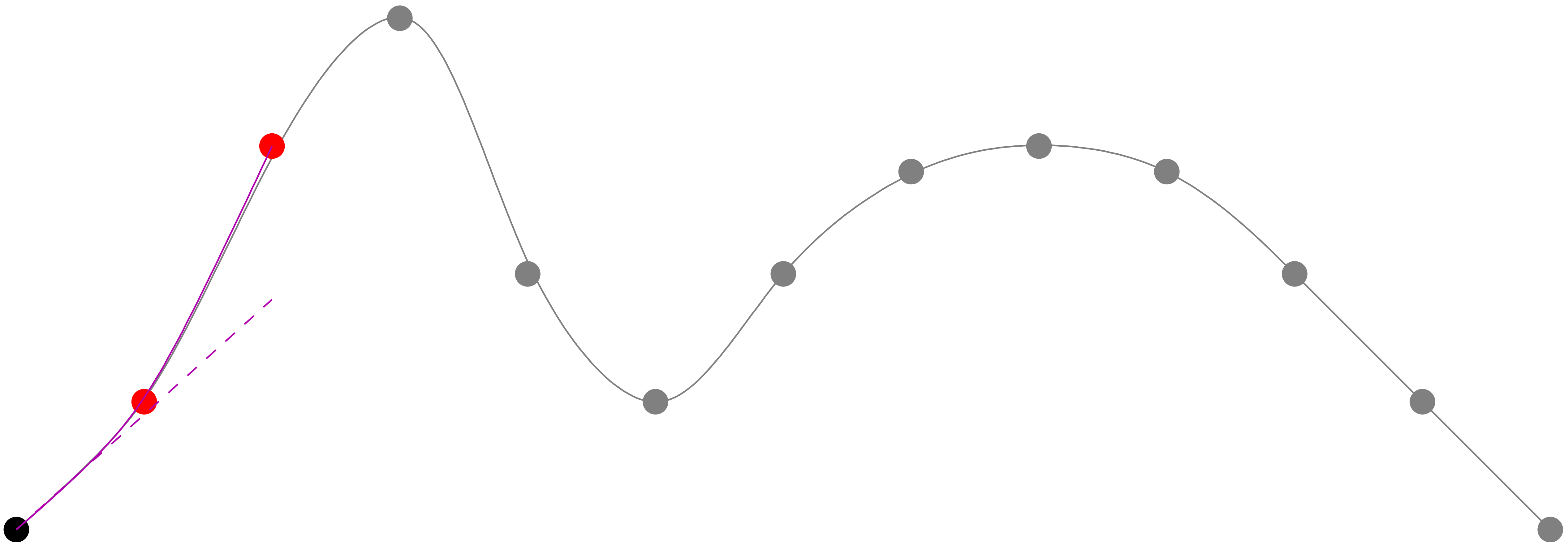

At a boundary:

$$f_j' = \frac{1}{\Delta x}\left(-\frac{3}{2}f_j + 2f_{j+1} - \frac{1}{2}f_{j+2}\right) = f'(x_j) + O\big(\Delta x^2\big).$$

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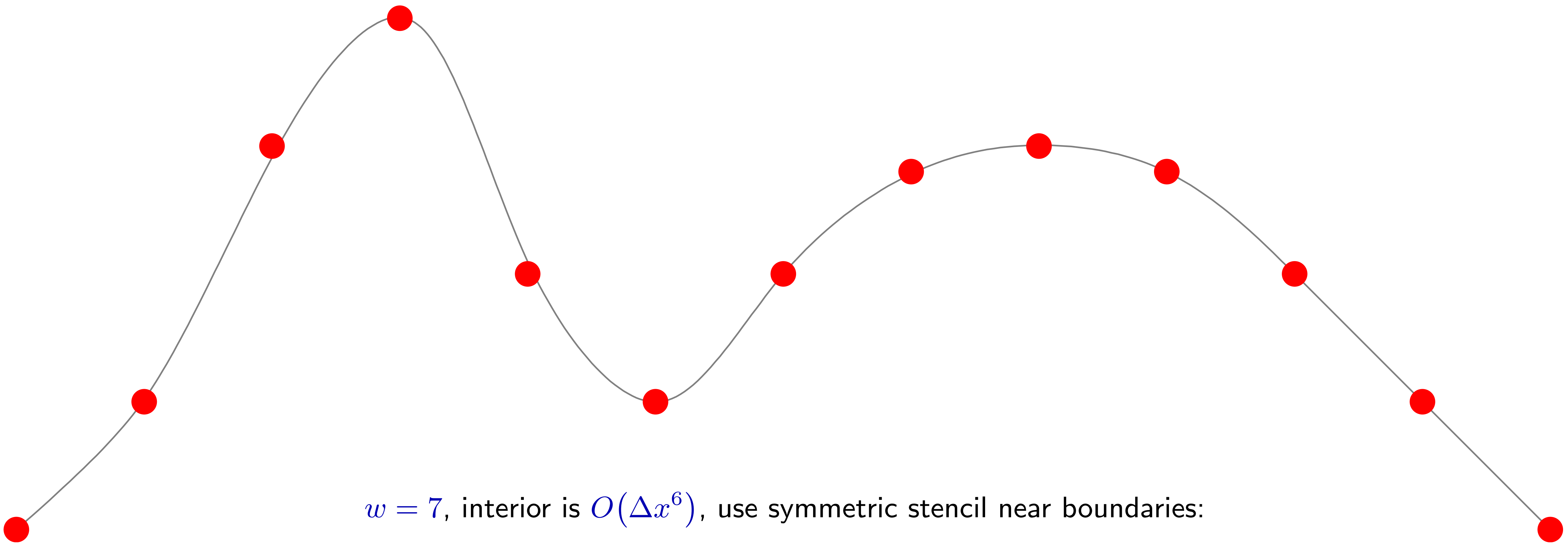

$w = 7$, interior is $O(\Delta x^6)$, use symmetric stencil near boundaries:

$$\begin{pmatrix} f_0' \\ f_1' \\ f_2' \\ f_3' \\ f_4' \\ \vdots \end{pmatrix} = \frac{1}{\Delta x} \begin{pmatrix} -3/2 & 2 & -1/2 & & & & & \\ -1/2 & 0 & 1/2 & & & & & \\ 1/12 & -2/3 & 0 & 2/3 & -1/12 & & & \\ -1/60 & 3/20 & -3/4 & 0 & 3/4 & -3/20 & 1/60 & \\ & -1/60 & 3/20 & -3/4 & 0 & 3/4 & -3/20 & 1/60 \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ \vdots \end{pmatrix}$$

Test case

- System to solve:

$$\frac{\partial p}{\partial t} + \frac{\partial v}{\partial x} = -\mu(x)p,$$
$$\frac{\partial v}{\partial t} + \frac{\partial p}{\partial x} = -\nu(x)v,$$

Boundary conditions: $v(0,t) = v(L,t) = 0$

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- There is an analytic solution (using characteristics). If $\mu \equiv \nu$, at time $t = 2L$,

$$p(x, 2L) = Ap(x, 0) \qquad v(x, 2L) = Av(x, 0) \qquad \text{were } A = \exp\left\{-2\int_0^L \mu(x)\,\mathrm{d}x\right\}$$

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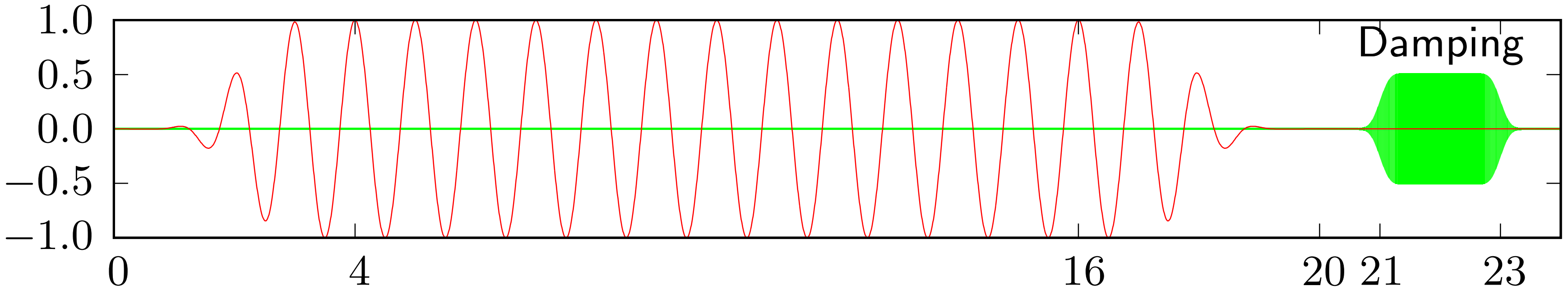
- Conserved laws:

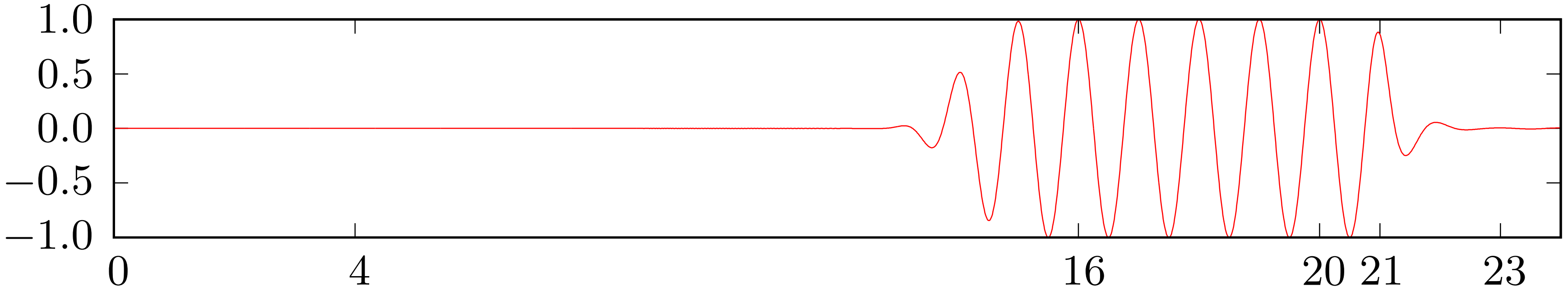
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L p\,\mathrm{d}x = -\int_0^L \mu p\,\mathrm{d}x, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\int_0^L v\,\mathrm{d}x = -\int_0^L \nu v\,\mathrm{d}x + p(0,t) - p(L,t),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \tfrac{1}{2}p^2 + \tfrac{1}{2}v^2\,\mathrm{d}x = -\int_0^L \mu p^2\,\mathrm{d}x - \int_0^L \nu v^2\,\mathrm{d}x.$$

Test case: 1D wave equation with damping

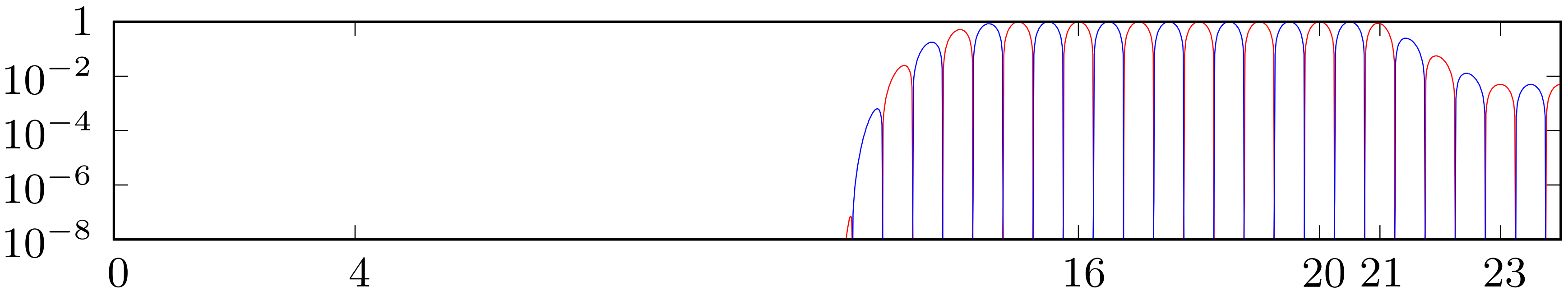
(a) $t = 0$



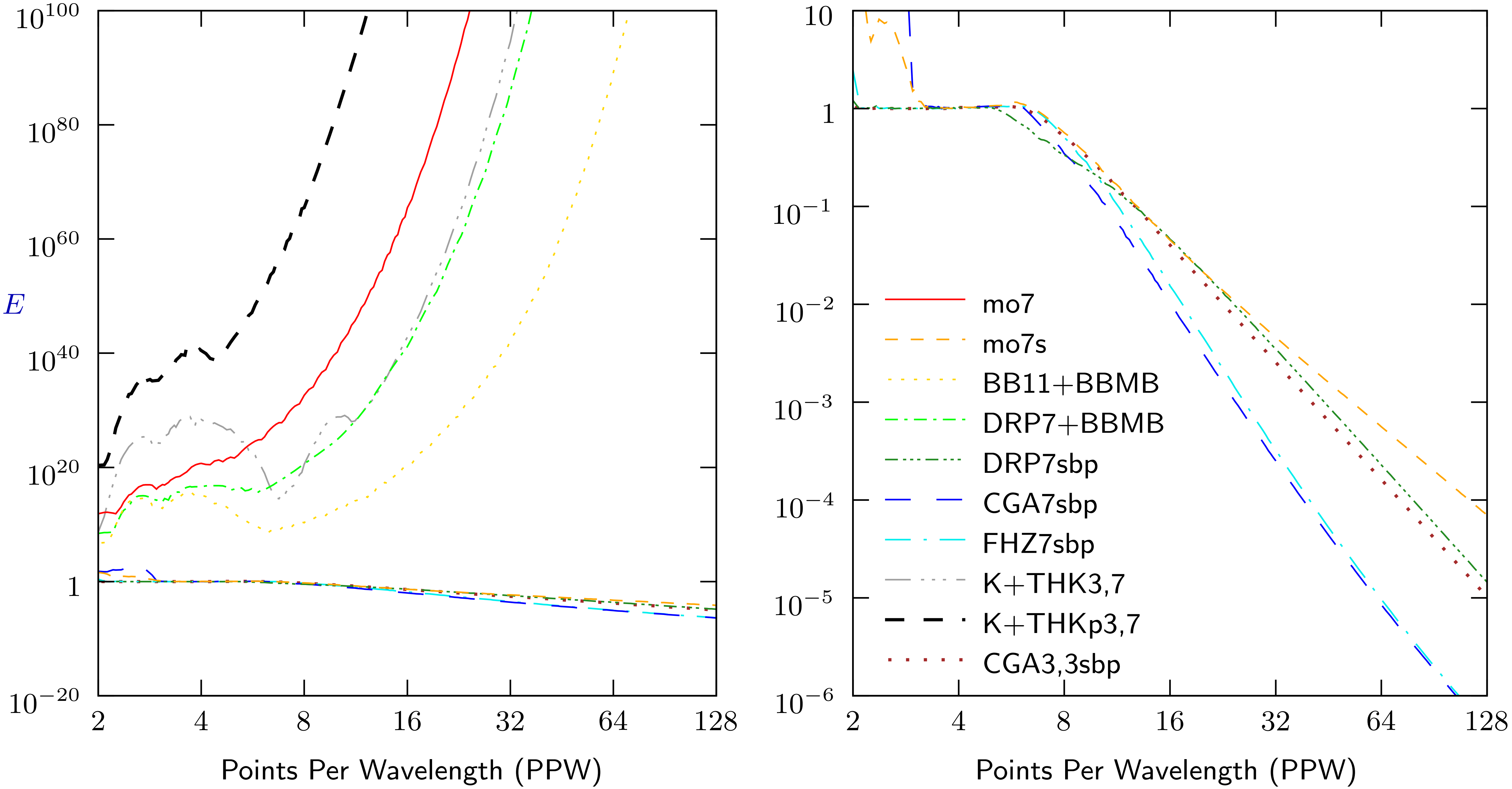
(b) $t = 12$



(c) $t = 12$ (logarithmic scale)



Test case performance



Summation by Parts (Strand, 1994)

- Integration by parts (IBP):

$$\langle f,\, g\rangle = \int_{x_0}^{x_N} f(x)g(x)\,\mathrm{d}x \qquad \Rightarrow \qquad \left\langle f,\, \frac{\mathrm{d}g}{\mathrm{d}x}\right\rangle = f(x_N)g(x_N) - f(x_0)g(x_0) - \left\langle \frac{\mathrm{d}f}{\mathrm{d}x},\, g\right\rangle$$

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- Discretize:

$$\boldsymbol{f} = (f_0, f_1, \ldots, f_N)^{\mathrm{T}} \qquad\qquad \boldsymbol{f}' = \mathsf{D}\boldsymbol{f}$$

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$$\langle f,\, g\rangle_{\mathsf{P}} = \boldsymbol{f}^{\mathbf{T}}\mathsf{P}\boldsymbol{g} = \sum_{i,j=1}^{N} f_i \mathsf{P}_{ij} g_j \qquad \Rightarrow \qquad \langle \boldsymbol{f},\, \mathsf{D}\boldsymbol{g}\rangle_{\mathsf{P}} = f_N g_N - f_0 g_0 - \langle \mathsf{D}\boldsymbol{f},\, \boldsymbol{g}\rangle_{\mathsf{P}}$$

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- Equivalently, $\mathsf{PD} = \mathsf{Q}$, with

$$\mathsf{P} = \mathsf{P}^{\mathrm{T}} \qquad\qquad \mathsf{Q} = -\mathsf{Q}^{\mathrm{T}} + \boldsymbol{e_N}\boldsymbol{e_N}^{\mathrm{T}} - \boldsymbol{e_0}\boldsymbol{e_0}^{\mathrm{T}}$$

Summation by Parts (4th order interior, 3rd order boundaries)

$$\Delta x \begin{pmatrix} \frac{173}{648} & \frac{41}{1296} & 0 & 0 & & & \\ \frac{41}{1296} & \frac{1135}{648} & \frac{-353}{648} & \frac{17}{108} & & & \\ 0 & \frac{-353}{648} & \frac{901}{648} & \frac{-151}{1296} & & & \\ 0 & \frac{17}{108} & \frac{-151}{1296} & \frac{671}{648} & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & \ddots \end{pmatrix} \begin{pmatrix} f_0' \\ f_1' \\ f_2' \\ f_3' \\ f_4' \\ f_5' \\ \vdots \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-1}{2} & \frac{2035}{2592} & \frac{-239}{648} & \frac{217}{2592} & & & & & \\ \frac{-2035}{2592} & 0 & \frac{829}{864} & \frac{-113}{648} & & & & & \\ \frac{239}{648} & \frac{-829}{864} & 0 & \frac{1747}{2592} & \frac{-1}{12} & & & & \\ \frac{-217}{2592} & \frac{113}{648} & \frac{-1747}{2592} & 0 & \frac{2}{3} & \frac{-1}{12} & & & \\ & & \frac{1}{12} & \frac{-2}{3} & 0 & \frac{2}{3} & \frac{-1}{12} & & \\ & & & \frac{1}{12} & \frac{-2}{3} & 0 & \frac{2}{3} & \frac{-1}{12} & \\ & & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ \vdots \end{pmatrix}$$

Summation by Parts and stability

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- Continuous conservation law:

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- Discrete conservation law:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{1}{2}\langle \boldsymbol{p},\boldsymbol{p}\rangle_{\mathsf{P}} + \tfrac{1}{2}\langle \boldsymbol{v},\boldsymbol{v}\rangle_{\mathsf{P}}\right) &= \left\langle \boldsymbol{p}, \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}\right\rangle_{\mathsf{P}} + \left\langle \boldsymbol{v}, \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right\rangle_{\mathsf{P}} \\ &= -\left\langle \boldsymbol{p}, \mathsf{D}\boldsymbol{v} + \mathsf{M}\boldsymbol{p}\right\rangle_{\mathsf{P}} - \left\langle \boldsymbol{v}, \mathsf{D}\boldsymbol{p} + \mathsf{N}\boldsymbol{v}\right\rangle_{\mathsf{P}} \\ &= -\langle \boldsymbol{p}, \mathsf{D}\boldsymbol{v}\rangle_{\mathsf{P}} - \langle \boldsymbol{p}, \mathsf{M}\boldsymbol{p}\rangle_{\mathsf{P}} + \langle \mathsf{D}\boldsymbol{v}, \boldsymbol{p}\rangle_{\mathsf{P}} - \langle \boldsymbol{v}, \mathsf{N}\boldsymbol{v}\rangle_{\mathsf{P}}\end{aligned}$$

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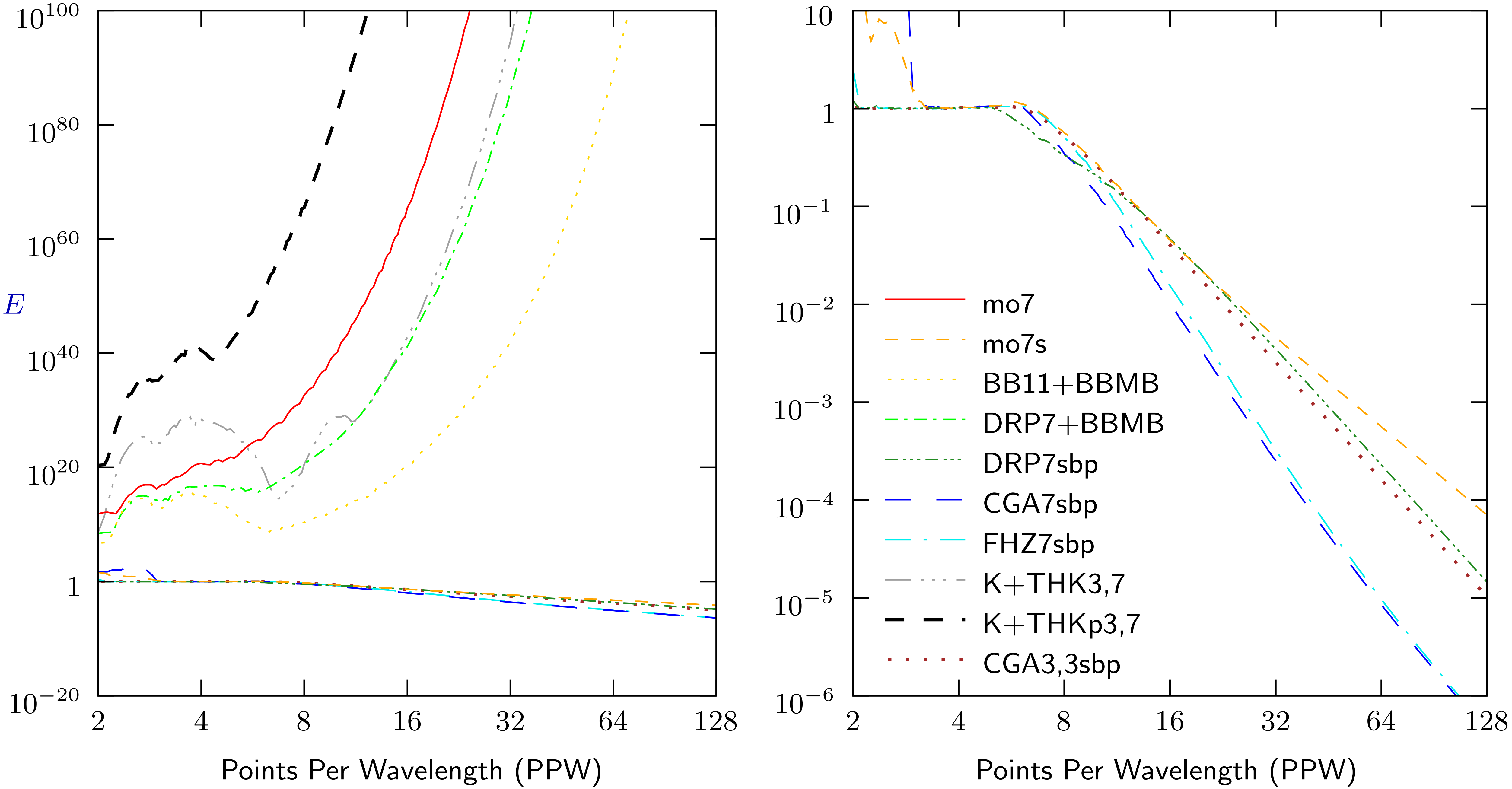

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then P and Q have the SBP property:

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$$\int_0^L \psi_i(x)\left(\hat{f}'(x) - \sum_j f_j'\psi_j(x)\right)\mathrm{d}x = 0 \qquad \forall i. \qquad \Rightarrow \qquad \mathsf{P}\boldsymbol{f}' = \mathsf{Q}\boldsymbol{f}$$

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  then P and Q have the SBP property:

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$$\boldsymbol{u}^T\mathsf{Q}\boldsymbol{v} = \int_0^{x_N} \hat{u}(x)\hat{v}'(x)\,\mathrm{d}x = [\hat{u}\hat{v}]_0^{x_N} - \int_0^{x_N} \hat{u}'(x)\hat{v}(x)\,\mathrm{d}x = u_N v_N - u_0 v_0 - \boldsymbol{v}^T\mathsf{Q}\boldsymbol{u},$$

- How do we differentiate $\hat{f}$ (as $\hat{f}'(x) = \sum_j f_j \psi_j'(x)$ is not necessarily in Ψ_N)?
- Pick $F(x) = \sum_j f_j' \psi_j(x)$ as the closest point in Ψ_N (using the L_2 norm) to $\hat{f}'(x) = \sum_j f_j \psi_j'(x)$;
  i.e. $F(x)$ is the orthogonal projection of $\hat{f}'(x)$ into Ψ_N, written $F = \mathcal{P}[\hat{f}']$, so

$$\int_0^L \psi_i(x)\left(\hat{f}'(x) - \sum_j f_j'\psi_j(x)\right)\mathrm{d}x = 0 \qquad \forall i. \qquad \Rightarrow \qquad \mathsf{P}\boldsymbol{f}' = \mathsf{Q}\boldsymbol{f}$$

- Hence, all interpolating bases naturally lead to an SBP derivative scheme. But of what order?

SBP Schemes from Interpolating Bases: Order

- Finite differences are order $O(\Delta x^p)$ at position i if it is exact for all polynomials of degree p,

$$\sum_j \mathsf{P}_{ij} n x_j^{n-1} = \sum_j \mathsf{Q}_{ij} x_j^n \qquad \forall n = 0, 1, \ldots, p.$$

- In interpolating basis notation,

$$\int_0^L \psi_i(x) \Big(\sum_j n x_j^{n-1} \psi_j(x) - \sum_j x_j^n \psi_j'(x) \Big) \, \mathrm{d}x = 0 \qquad \forall n = 0, 1, \ldots, p.$$

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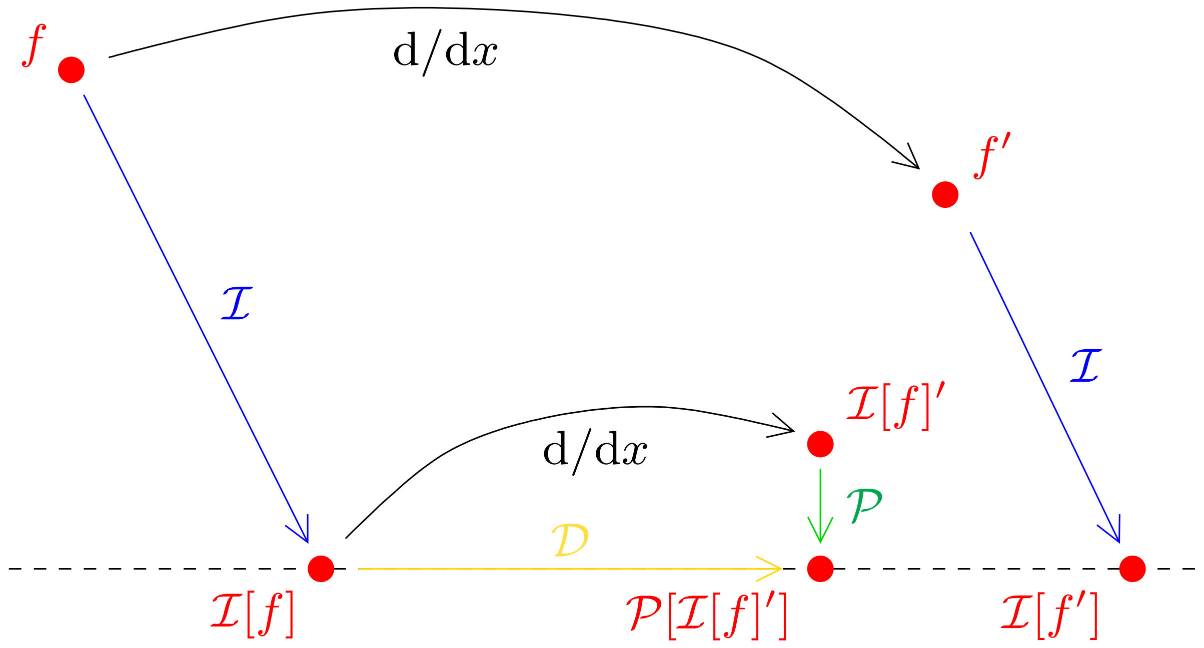
- If this is true for all i, then $\mathcal{I}[P'] = \mathcal{P}[\mathcal{I}[P]']$ for polynomials $P(x)$ up to degree p:



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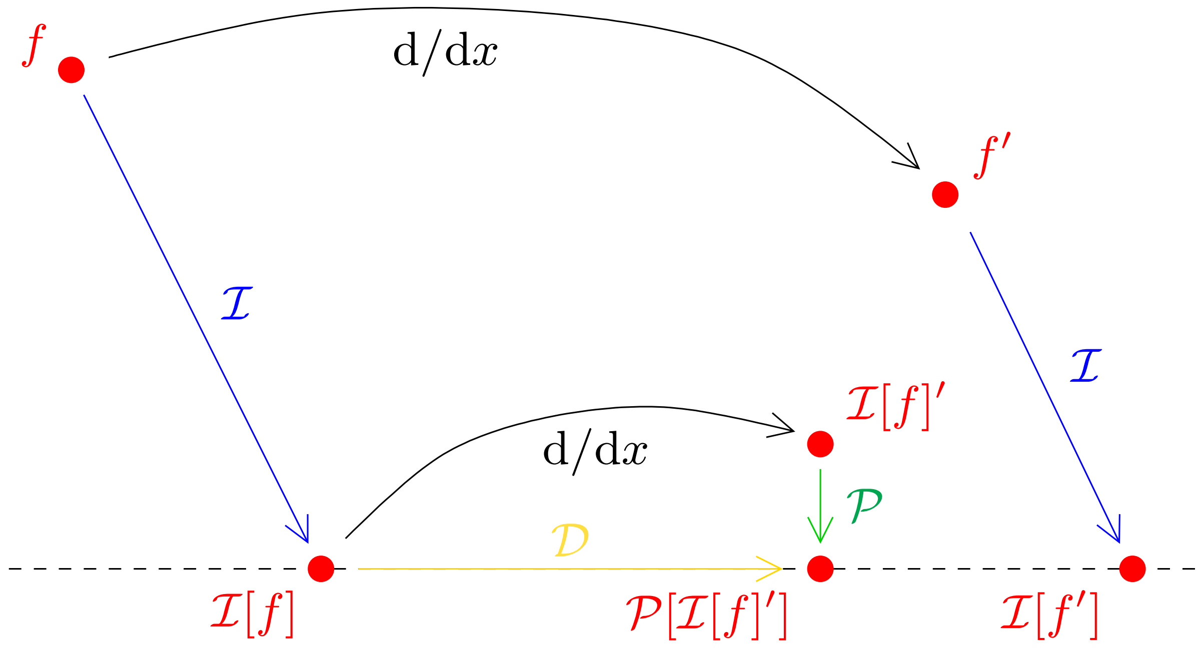

- This is trivially true if $\mathcal{I}P = P$ for polynomials of degree p.

An example: Piecewise Linear Interpolation

- Take basis functions corresponding to piecewise linear interpolation,

$$\psi_j(x) = \begin{cases} 1 - |x - x_j| & |x - x_j| < 1 \\ 0 & |x - x_j| \geq 1 \end{cases}$$

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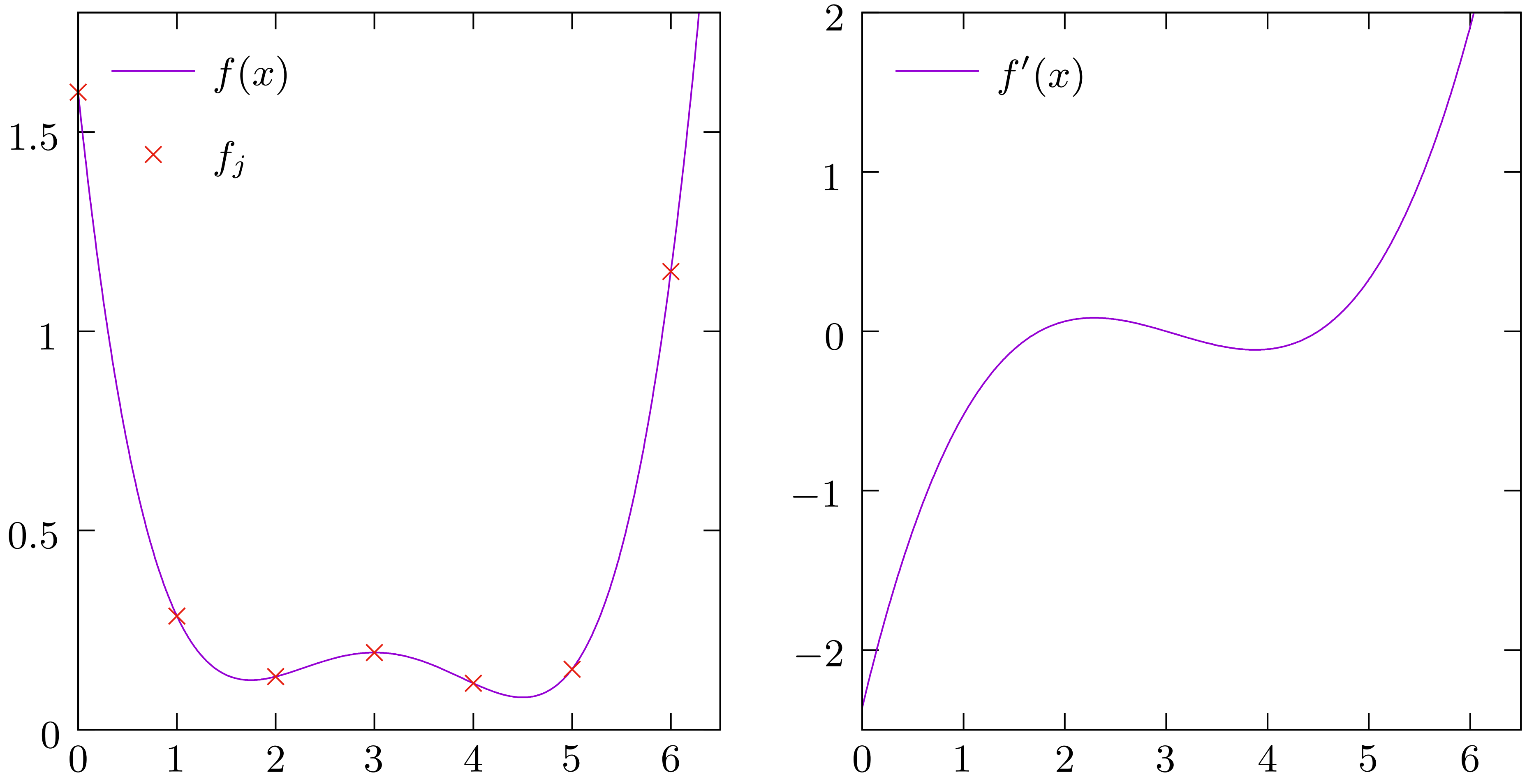

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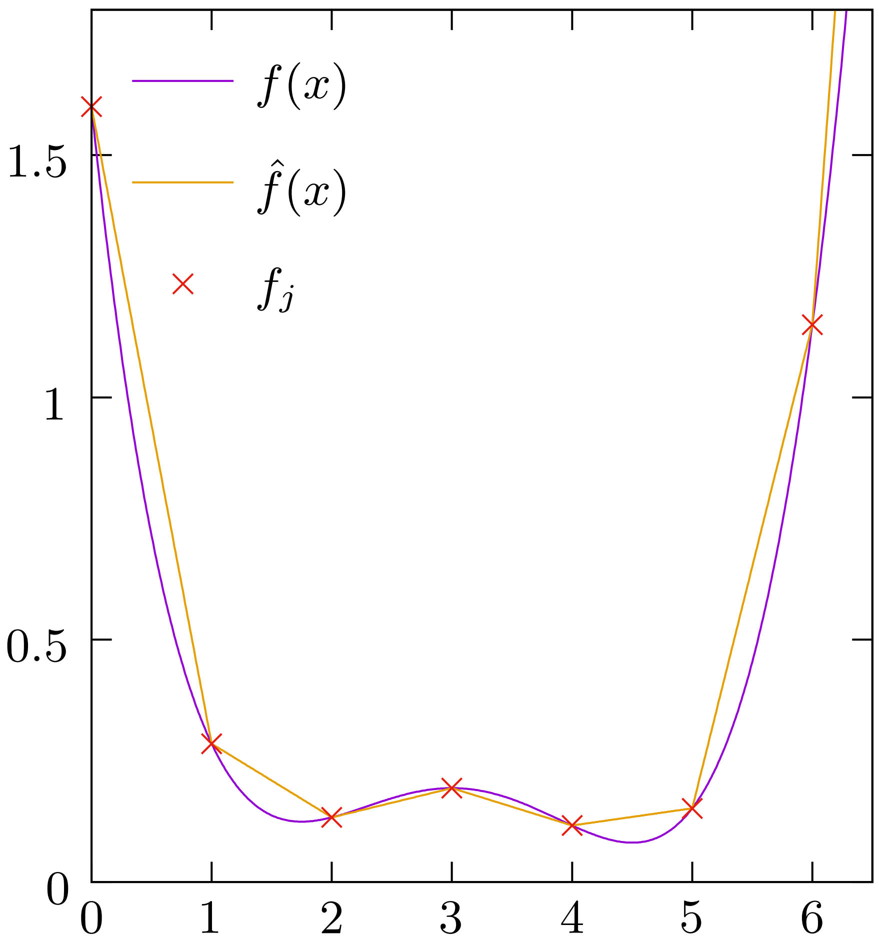

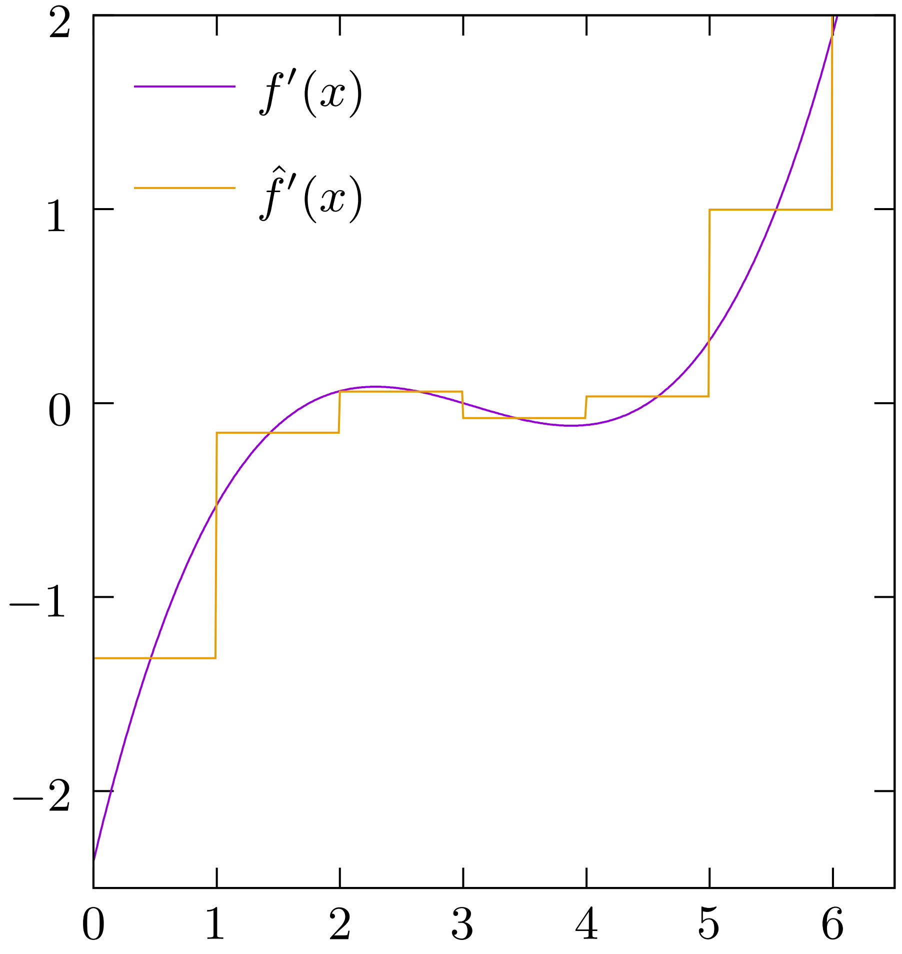

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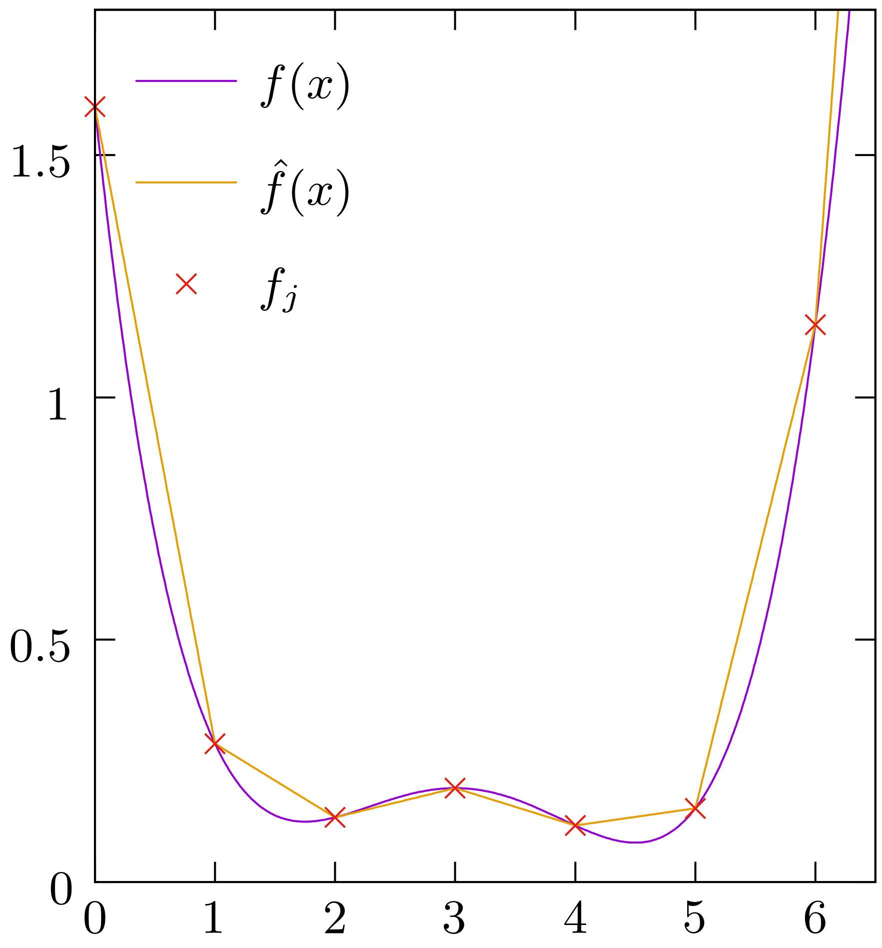

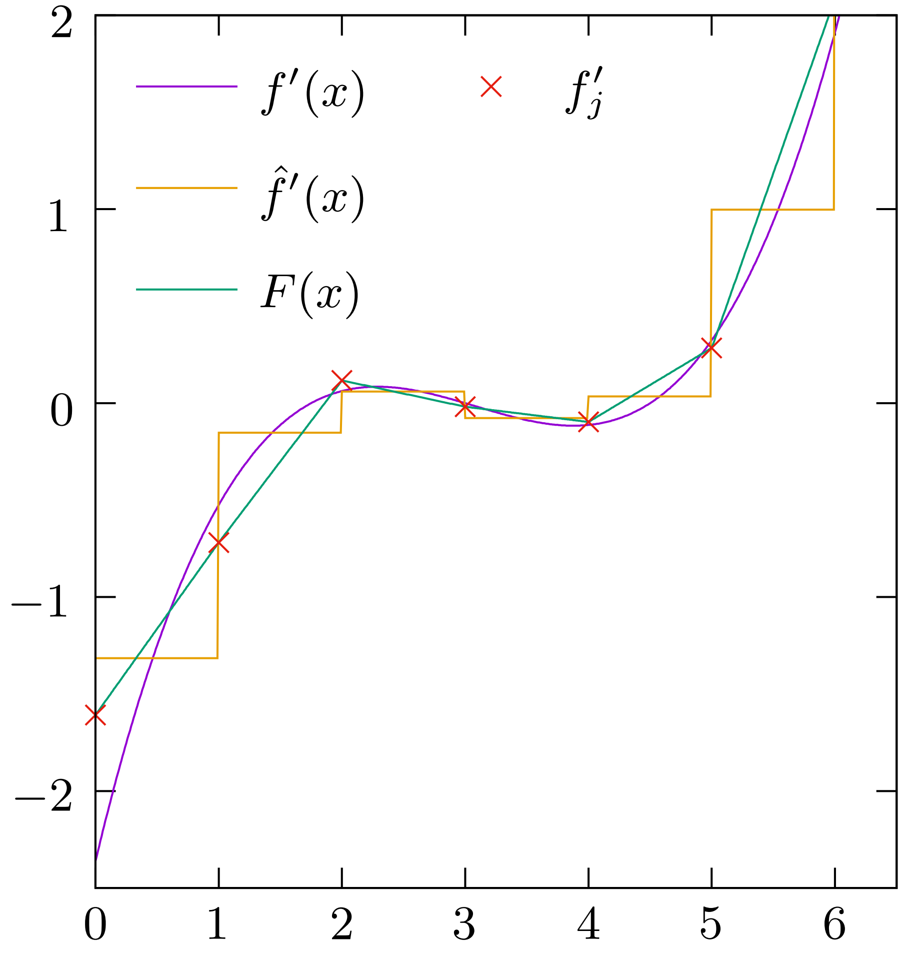

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- By calculation, we find

$$\mathsf{P}_{ij} = \int_0^L \psi_i(x)\psi_j(x)\,\mathrm{d}x = \begin{pmatrix} \frac{1}{3} & \frac{1}{6} & & & \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & & \\ & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \quad \mathsf{Q}_{ij} = \int_0^L \psi_i(x)\psi_j'(x)\,\mathrm{d}x = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & & & \\ -\frac{1}{2} & 0 & \frac{1}{2} & & \\ & -\frac{1}{2} & 0 & \frac{1}{2} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

  - This is a 4th order tri-diagonal maximal order scheme in the interior.
  - It is only 1st order at the boundary.
  - The interpolation is only exact for polynomials of degree 1.